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On the position uncertainty measure on the circle

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Abstract
New position uncertainty (delocalization) measures for a particle on the circle are proposed and illustrated in several examples, where the previous measures (based on 2π-periodic position operators) appear to be unsatisfactory. The new measures are suitably constructed using the standard multiplication angle operator variances. They are shown to depend solely on the state of the particle and to obey uncertainty relations of the Schrödinger–Robertson type.

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1. Introduction

Recently there is a renewed interest to the old problem of the uncertainties and uncertainty relations for a particle on the circle [1–6]. Due to the controversial commutation relation between the angle and angular momentum operators most attention has been paid to position operators that are invariant under translation $\varphi \longrightarrow \varphi + a$, $a \in \mathbb{R}$. The relation $[\hat{l}_z, \hat{\varphi}] = -\mathrm{i}$ stems from the Dirac correspondence rule

$$\{f, g\} \longrightarrow \mathrm{i}[\hat{f}, \hat{g}] \tag{1}$$

between Poisson bracket $\{f, g\}$ of two classical quantities f and g and the commutator of the corresponding operators $\hat{f}$ and $\hat{g}$ for quantum observables. This rule is formally satisfied with

$$\hat{\varphi} = \varphi \qquad \text{and} \qquad \hat{l}_z = -\mathrm{i}\,\mathrm{d}/\mathrm{d}\varphi. \tag{2}$$

However, on the eigenstates

$$\psi_m(\varphi) = \exp(\mathrm{i}m\varphi)/\sqrt{2\pi} \qquad m = 0, \pm 1, \ldots \tag{3}$$

the above commutation relation breaks down, together with the associated standard Heisenberg–Robertson uncertainty relation $(\Delta l_z)^2(\Delta\varphi)^2 \geqslant 1/4$. Therefore authors try to adopt another position operator [4, 5, 7], or even another definition of the uncertainty on the circle [3].

In this paper, we provide an approach to the issue with minimal (in our opinion) deviation from the standard commutation relation and standard measure of uncertainty. The main idea has been sketched in [2]. Here we develop it in greater detail, providing some proofs and further examples. After a brief review of the properties of main previous position uncertainty measures in section 2, two different new measures are constructed and discussed in section 3. The new measures are constructed using suitably the standard expressions of the first and second moments of the angle variable, calculated by integration over 2π intervals. They are of the form of positive state functionals, the values of which depend solely on the state considered.

The terms position uncertainty measure and state delocalization measure are used here as synonyms. The uncertainty measure of states is also called the measure of spread of corresponding wavefunctions (more precisely of the corresponding probability distributions $p(\varphi) = |\psi(\varphi)|^2$). It is worth noting that all uncertainty measures are maps of the infinite-dimensional state space into the positive part of the real line. It is impossible in such a way to distinguish between all states. Therefore different measures should be considered not only as competitive, but as complementary as well.

2. A brief review of previous delocalization measures

For a particle on the real line the standard measure of the position uncertainty is given by the second moment $(\Delta x)^2 := \langle (x - \langle x\rangle)^2\rangle$ of the position operator $\hat{x} = x$, or equivalently by the standard deviation Δx. Mathematically both Δx and $\langle x\rangle$ are one-to-one functionals on state space. The quantity $(\Delta x)^2$ is also called variance, or dispersion, of x and is also denoted as Dx or $M^{(2)}x$. The variance of x is regarded as a measure of spread, or delocalization, of the state wavefunction $\psi(x)$. More precisely, this is a measure of spread of the probability distribution $p(x) := |\psi(x)|^2$. Here the means $\langle x\rangle$ and $\langle x^2\rangle$ are calculated by integration with respect to x: $\langle x\rangle = \int x|\psi(x)|^2\,\mathrm{d}x$.

However, in the case of angle operator $\hat{\varphi} = \varphi$ it was not clear how to calculate and interpret the analogous quantity $\Delta\varphi$, since the operator $\hat{\varphi} = \varphi$ is not invariant under translation $\varphi \to \varphi + 2\pi$ (not 2π-periodic), while the wavefunctions $\psi(\varphi)$ are 2π-periodic by definition. This trouble seems to be the main reason why many authors look for 2π-invariant position operators in order to construct relevant uncertainty measures on the circle.

The first such operators used probably were $\sin\varphi$ and $\cos\varphi$ [7]. The variances of these operators satisfy correct inequalities [7]

$$(\Delta l_z)^2(\Delta\sin\varphi)^2 \geqslant |\langle\cos\varphi\rangle|^2/4 \qquad (\Delta l_z)^2(\Delta\cos\varphi)^2 \geqslant |\langle\sin\varphi\rangle|^2/4. \tag{4}$$

However, one can see that the variances $(\Delta\sin\varphi)^2$ and $(\Delta\cos\varphi)^2$ may take values greater than the corresponding one for the uniform distribution $p_0(\varphi) = 1/2\pi = |\psi_m(\varphi)|^2$; in $\psi_m(\varphi)$ one has $(\Delta\sin\varphi)^2 = (\Delta\cos\varphi)^2 = 1/2$), in $\psi_c(\varphi) = (1/\sqrt{\pi})\cos\varphi$ one has $(\Delta\cos\varphi)^2 = 3/4$, $(\Delta\sin\varphi)^2 = 1/4$, and in $\psi_s(\varphi) = (1/\sqrt{\pi})\sin\varphi$ these are interchanged—$(\Delta\cos\varphi)^2 = 1/4$, $(\Delta\sin\varphi)^2 = 3/4$. The two states $\psi_c(\varphi)$ and $\psi_s(\varphi)$ coincide under the shift $\varphi \to \varphi \pm \pi/2$, therefore it is reasonable to have coinciding (or close) measures of spread for them, which should be less than those in the eigenstates $\psi_m(\varphi)$. These deficiencies are partially removed by the 'uncertainty measure' [7] $(\tilde{\Delta}\varphi)^2 = (\Delta\cos\varphi)^2 + (\Delta\sin\varphi)^2$, which can also be written in the forms

$$(\tilde{\Delta}\varphi)^2 = 1 - \langle\cos\varphi\rangle^2 - \langle\sin\varphi\rangle^2 = 1 - |\langle U(\varphi)\rangle|^2 \qquad U(\varphi) = \mathrm{e}^{\mathrm{i}\varphi}. \tag{5}$$

The quantity $\tilde{\Delta}\varphi$ has been considered also in [4, 5]. In [4] it was noted that $\tilde{\Delta}\varphi$ has the meaning of radial distance of the centroid of the ring distribution $p(\varphi)$ from the circle line (and $\langle\cos\varphi\rangle^2 + \langle\sin\varphi\rangle^2$ is the squared centroid's distance from the centre of the circle—see figure 1 in [4]). From (5) and (4) it follows that [7]

$$(\Delta l_z)^2(\tilde{\Delta}\varphi)^2 \geqslant \tfrac{1}{4}(\langle\cos\varphi\rangle^2 + \langle\sin\varphi\rangle^2). \tag{6}$$

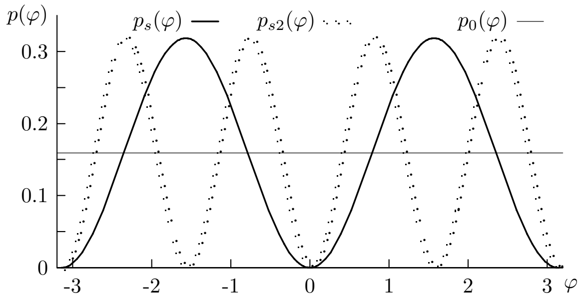

Figure 1. π-periodic, $\pi/2$-periodic and uniform distributions on the circle $p_s(\varphi) = |\psi_s(\varphi)|^2$, $p_{s2}(\varphi) = |\psi_{s2}(\varphi)|^2$ and $p_0(\varphi) = |\psi_m(\varphi)|^2$. The functional $\tilde{\Delta}\varphi$, equation (5), on all these distributions takes the same maximal value of 1, while $\Delta^2(\hat{\varphi})$, equation (7), takes the values 0.346, ∞ and ∞ respectively.

This uncertainty relation is approximately minimized in the canonical coherent states (CS) $|\alpha, \beta\rangle$ of the two-dimensional oscillator with large value of $\mathrm{Re}^2\alpha + \mathrm{Re}^2\beta$ [7].

However, if one considers the quantity $(\tilde{\Delta}\varphi)^2$, equation (5), as a delocalization measure on the circle one encounters some unsatisfactory results. For example, it produces the same maximal delocalization (i.e. $\tilde{\Delta}\varphi = 1$) for the eigenstates $\psi_m(\varphi)$ of $\hat{l}_z$ and for all states $\psi(\varphi)$ with the property $|\psi(\varphi)| = |\psi(\varphi + \pi)|$. The centroid for those π-periodic distributions $|\psi(\varphi)|^2$ is in the centre of the ring. In figure 1 graphics of three π-periodic distributions are shown: the uniform one $p_0(\varphi) = 1/2\pi = |\psi_m(\varphi)|^2$, $p_s(\varphi) = |\psi_s(\varphi)|^2 = \sin^2\varphi/\pi$ and $p_{s2}(\varphi) = |\sin(2\varphi)|^2/\pi$. It is clear that the localization of those distributions is quite different, and it is desirable to have an uncertainty measure that distinguishes between them.

Rather nonstandard expressions for position and angular momentum uncertainties for a particle on the circle were introduced and discussed in [3]:

$$\Delta^2(\hat{l}_z) = \tfrac{1}{4}\ln(\langle \mathrm{e}^{-2\hat{l}_z}\rangle\langle \mathrm{e}^{2\hat{l}_z}\rangle) \qquad \Delta^2(\hat{\varphi}) = -\tfrac{1}{4}\ln|\langle U(\varphi)^2\rangle|^2. \tag{7}$$

For a large set of states these quantities obey the inequality $\Delta^2(\hat{l}_z) + \Delta^2(\hat{\varphi}) \geqslant 1$, the equality being reached in the eigenstates $|\xi\rangle$ of the operator $Z = \exp(-\hat{l}_z + 1/2)U(\varphi)$. The family of $|\xi\rangle$ is overcomplete and the states $|\xi\rangle$ are called CS on the circle [3, 5, 6, 8].

The functional $\Delta^2(\hat{\varphi})$ was proposed as a position uncertainty on the circle. However, this uncertainty measure was found [2] to be not quite consistent with state localization: on CS $|\xi\rangle$ it equals 1/2, while on the visually worse localized states $|\xi\rangle - |-\xi\rangle$ (Schrödinger cat states on the circle) it can take the smaller rather value of 0.33 (see [2] and figure 2 therein). On the above noted states $\psi_s(\varphi)$, $\psi_{s2}(\varphi)$ and $\psi_m(\varphi)$ it takes values 0.346, ∞, ∞. Thus it makes a distinction between $\psi_s(\varphi)$ and $\psi_{s2}(\varphi)$ and $\psi_m(\varphi)$, but identifies $\psi_{s2}(\varphi)$ with the uniform state $\psi_m(\varphi)$ (see figure 1). Another unsatisfactory property of $\Delta^2(\hat{\varphi})$ is that it takes the smaller value of 0.143 on the two-peak state $\psi_{s4}(\varphi) = (0.2 + \sin^2\varphi)^2/N$, while on the CS $|z = 1\rangle$ (one-peak state) it assumes the much larger value of 0.5 (see figure 2).

3. Generalized uncertainty measures based on the variance

The state space of a particle on the circle consists of 2π-periodic square-integrable functions $\psi(\varphi)$. (In fact periodicity is up to a phase factor.) In view of this periodicity the scalar product of two states $\psi_1(\varphi)$ and $\psi_2(\varphi)$ can be calculated by integration with respect to φ within any interval of length 2π. Since $\varphi\psi(\varphi)$ is no longer periodic in φ the standard second moment

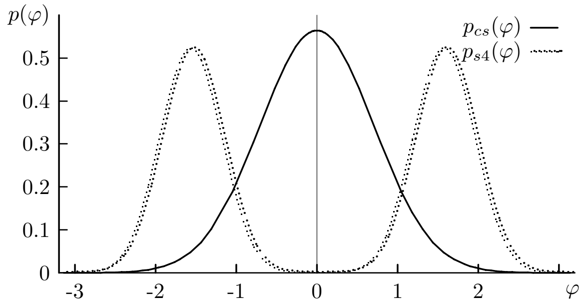

Figure 2. One- and two-peak φ-distributions $p_{cs}(\varphi)$, $p_{s4}(\varphi)$, corresponding to the CS $|\xi = 1\rangle$ and to state $\psi_{s4}(\varphi) = \text{const}\,(0.2 + \sin^2\varphi)^2$ on the circle. Here $\tilde{\Delta}_{p_{cs}}\varphi < \tilde{\Delta}_{p_{s4}}\varphi = 1$, while $\Delta^2_{p_{cs}}(\hat{\varphi}) > \Delta^2_{p_{s4}}(\hat{\varphi}) = 0.143$.

$D\varphi \equiv (\Delta\varphi)^2$ of φ would naturally depend on the interval of integration (here specified by the reference point φ_0),

$$D\varphi = \int_{\varphi_0-\pi}^{\varphi_0+\pi} \left(\varphi - \langle\varphi\rangle_{\varphi_0}\right)^2 |\psi(\varphi)|^2\, \mathrm{d}\varphi = D\varphi(\varphi_0) \tag{8}$$

$$\langle\varphi\rangle_{\varphi_0} = \int_{\varphi_0-\pi}^{\varphi_0+\pi} \varphi |\psi(\varphi)|^2\, \mathrm{d}\varphi = M\varphi(\varphi_0). \tag{8a}$$

This φ_0-dependence of the standard moments of φ is the main reason why authors abandon $D\varphi$ and look for other expressions to simulate quantum position uncertainties on the circle or, equivalently, the spread of the related periodic probability distributions $p(\varphi)$. It turns out, however, that the variance (8) could still be useful in construction of relevant uncertainty measures. Expressions of the type $\int_{\theta-\pi}^{\theta+\pi} (\phi - \theta)^2 p(\phi)\, \mathrm{d}\phi$ have been examined as measures of optical phase resolution in [10]. Let us note that $\langle\varphi\rangle_{\varphi_0}$ and $D\varphi(\varphi_0)$ are treated correctly here as functionals, aiming to construct a relevant delocalization measure. The full problem of the correct Hermitian position operator on the circle deserves special and separate consideration.

First of all we note that if one defines the φ_0-dependent covariance $\Delta l_z\varphi(\varphi_0)$ of $\hat{\varphi}$ and $\hat{l}_z$ as the real part of the matrix element $G_{l_z\varphi} := \langle(\hat{l}_z - \langle\hat{l}_z\rangle)\psi|(\hat{\varphi} - \langle\varphi\rangle)\psi\rangle$, $\Delta l_z\varphi(\varphi_0) = \mathrm{Re}\, G_{l_z\varphi}(\varphi_0)$, (where the means are taken by integration as in (8)), one obtains the inequality (see also [2, 9])

$$D\varphi(\varphi_0) Dl_z - (\Delta l_z\varphi(\varphi_0))^2 \geqslant \left(\mathrm{Im}\, G_{l_z\varphi}(\varphi_0)\right)^2 \tag{9}$$

which is a generalization of the Schrödinger (or Schrödinger–Robertson) uncertainty relation [11]. For a particle on the real line the latter relation reads $DxDp - (\mathrm{Cov}(x, p))^2 \geqslant 1/4$, where $\mathrm{Cov}(x, p) \equiv \Delta xp$ is the covariance of $\hat{x}$ and $\hat{p}$. The problem remains however to define on the circle uncertainty (or delocalization, or spread) measure $\Delta^2_{|\psi\rangle}\varphi$ of the state $|\psi\rangle$ (or of the distribution $p(\varphi)$) that *depends solely on the state* $|\psi\rangle$, and not on the limit of integration in (8). It turned out that this problem can be resolved by a suitable use of $D\varphi(\varphi_0)$ due to the 2π-periodic property of the functional (8),

$$D\varphi(\varphi_0 + 2\pi) = D\varphi(\varphi_0). \tag{10}$$

The property (10) can be easily proved, using the state periodicity $|\psi(\varphi + 2\pi)| = |\psi(\varphi)|$ and the definition of $D\varphi(\varphi_0)$. In fact one can show that all moments $M^{(n)}\varphi(\varphi_0) = \langle(\varphi - \langle\varphi\rangle)^n\rangle$, $n = 1, \ldots,$ of φ are 2π-periodic in φ_0. In view of this periodic property the φ_0-independent uncertainty measure can be defined in two different ways:

(a) as an arithmetic mean of $D\varphi(\varphi_0)$ with respect to $\varphi_0 \in I_{2\pi}$, and
(b) as an extremal value[1] of $D\varphi(\varphi_0)$ in $I_{2\pi}$, where $I_{2\pi}$ is any interval of length 2π,

$$\text{(a)} \quad {}_a\Delta^2\varphi = \frac{1}{2\pi}\int_{I_{2\pi}} D\varphi(\varphi_0)\,\mathrm{d}\varphi_0 \tag{11}$$

$$\text{(b)} \quad {}_b\Delta^2\varphi = \underset{\varphi_0\in I_{2\pi}}{\text{minimum}}\ D\varphi(\varphi_0). \tag{12}$$

We introduce also the arithmetic mean squared covariance (by integration in any 2π interval $I_{2\pi}$)

$${}_a(\Delta l_z\varphi)^2 = \frac{1}{2\pi}\int_{I_{2\pi}} (\Delta l_z\varphi(\varphi_0))^2\,\mathrm{d}\varphi_0. \tag{13}$$

Then taking into account equations (9), (11)–(13) and the fact that minimum of $D\varphi(\varphi_0)$ is achieved at some $\varphi_0 = \varphi_{\min}$, we arrive at two Schrödinger-type uncertainty relations $(\Delta^2 l_z = Dl_z = (\Delta l_z)^2)$

$${}_i\Delta^2\varphi\Delta^2 l_z - {}_i(\Delta l_z\varphi)^2 \geqslant {}_i\big(\mathrm{Im}\,G_{l_z\varphi}\big)^2 \tag{14}$$

where $i = a, b$ and ${}_a\big(\mathrm{Im}\,G_{l_z\varphi}\big)^2$ is the arithmetic mean of $\big(\mathrm{Im}\,G_{l_z\varphi}(\varphi_0)\big)^2$. Thus both measures ${}_a\Delta\varphi$ and ${}_b\Delta\varphi$ are supported by inequalities of the type of Schrödinger uncertainty relation. It follows from this analogy that the quantities ${}_i\Delta^2\varphi$, $\Delta^2 l_z$ and ${}_i(\Delta l_z\varphi)^2$ could be regarded as (generalized) second moments of $\hat{\varphi}$ and $\hat{l}_z$.

The examinations show that in a variety of examples the quantities ${}_a\Delta^2\varphi$ and ${}_b\Delta^2\varphi$ behave as relevant position uncertainty measures on the circle. Both measures distinguish between all states presented in figures 1 and 2, their value for the uniform distribution being greater than that for the other distributions. On the states in figures 1 and 2 we have a satisfactory arrangement of the spread measures, consistent with the visualized localization. The values of ${}_b\Delta^2\varphi$, for example, read

$${}_b\Delta^2\varphi|_{p_0(\varphi)} = \frac{\pi^2}{3} > {}_b\Delta^2\varphi|_{p_{s2}(\varphi)} = 3.16 > {}_b\Delta^2\varphi|_{p_s(\varphi)} = 2.79 \qquad \text{(figure 1)} \tag{15}$$

$$\frac{\pi^2}{3} > {}_b\Delta^2\varphi|_{p_{s4}(\varphi)} = 2.61 > {}_b\Delta^2\varphi|_{p_{cs}(\varphi)} = 0.5 \qquad \text{(figure 2).} \tag{16}$$

Compare the results (15) and (16) with the corresponding values of measures (5) and (7). For example, compare (16) with $\Delta^2(\hat{\varphi})|_{p_{s4}} = 0.346 < \Delta^2(\hat{\varphi})|_{p_{cs}} = 0.5$.

There is a third invariantly defined state characteristic point on the circle (the first two are the points where $D\varphi(\varphi_0)$ attains its extrema). This third point is the *centre of the packet* $p(\varphi)$, denoted here as φ_c. For a large set of distributions the centre of the packet φ_c can be defined and determined as the angle of the centroid of $p(\varphi)$. The Cartesian coordinates of the centroid are $x = \langle\cos\varphi\rangle$ and $y = \langle\sin\varphi\rangle$. We define the third measure of spread of $p(\varphi)$ as ${}_c\Delta^2\varphi$ [2],

$${}_c\Delta^2\varphi = D\varphi(\varphi_0 = \varphi_c) \tag{17}$$

where $D\varphi(\varphi_0)$ is the second moment (8).

The choice of $\varphi_0 = \varphi_c$ in the limits of integration in (8) was wrongly interpreted in [1] as the introduction of a *definition* of average values depending on the particular state. To reveal this misinterpretation suffice it to recall that φ_c is a characteristic point of the distribution $p(\varphi)$, therefore of the state $\psi(\varphi)$: the value of φ_c, and thereby the value of ${}_c\Delta^2\varphi$ and $\langle\varphi\rangle_{\varphi_c}$

[1] We consider the minimal value only, since the maximal one may be greater than that of the uniform distribution.

are determined solely by the state $|\psi\rangle$. Thus ${}_c\Delta^2\varphi$, first proposed in [2], is a correct positive functional of the state and may be examined as an uncertainty measure.

A problem with the definition (17) appears in the case of π-periodic distributions $p(\varphi)$, since in such cases the centroid angle is not determined (the centroid is at the origin). The difficulty however is easily overcome if one notes [4] that *the centroid is a natural measure of the mean of the distribution*. This gives a hint to define more generally the centre of the packet φ_c as the solution of the equation

$$M\varphi(\varphi_0) = \varphi_0 \tag{18}$$

where $M\varphi(\varphi_0)$ is the limit-dependent mean of φ given by ($8a$). The examination shows that the centroid angle φ_c, when it exists, is a solution of equation (18). For π-periodic distributions the centroid is at the origin, and φ_c remains undefined. It turned out that for such distributions equation (18) has more than one solution, i.e. there are several equivalent points $\varphi_{c,i}$. We will say that in such cases several points $\varphi_{c,i}$ on the circle may serve as 'centres of the packet', or the packet is *'multi-centred'*. If $p(\varphi+\pi/k) = p(\varphi)$, $k = 1,\ldots,n$ then equation (18) should have $2n$ different solutions $\varphi_{c,i}$, $i = 1,\ldots,2n$. For $p_s(\varphi)$, $p_{2s}(\varphi)$ in figure 1 (and $p_{cs}(\varphi)$, $p_{s4}(\varphi)$ in figure 2) we have solutions $\varphi_c = \pm\pi/2$, $\varphi_c = \pm\pi/4, \pm 3\pi/4$ (and $\varphi_c = 0$, $\varphi_c = 0, \pi$). For the uniform distribution equation (18) degenerates to the identity $\varphi_0 = \varphi_0$, i.e. for $p_0(\varphi)$ all points on the circle are equivalent.

Equation (18) may be difficult for analytical handling, but solutions can be easily found numerically, or by the following rule/ansatz: $\varphi_{c,i}$ are points $\varphi_{\min}$ of the global minimum of the second moment $D\varphi(\varphi_0)$, equation (8), as a function of φ_0. This means that $\Delta\varphi(\varphi_{c,i})$, $i = 1,\ldots,n$, coincide, and

$$D\varphi(\varphi_{c,i}) = {}_b\Delta^2\varphi \qquad i = 1,\ldots,n. \tag{19}$$

The rule works (is confirmed) in the example of a variety of distributions $p(\varphi)$, in particular in all examples in figures 1 and 2. Since the global minimum of $D\varphi(\varphi_0)$ can be calculated invariantly in any interval $I_{2\pi} \ni \varphi_0$ the above coincidence confirms again that the measure ${}_c\Delta^2\varphi$ depends solely on the state. It was shown in the first paper of [10] that the condition $\bar{\phi} = \int_{\bar{\phi}-\pi}^{\bar{\phi}+\pi} \phi p(\phi)\,\mathrm{d}\phi$, where $p(\phi)$ is the canonical phase distribution, is a necessary one for the minimum of the variance $\int_{\theta-\pi}^{\theta+\pi} (\phi-\theta)^2 p(\phi)\,\mathrm{d}\phi$.

4. Conclusion

In this paper, we have introduced and discussed new position uncertainty (delocalization) measures for a particle on the circle. The relevant measure properties are illustrated in several examples, where the previous measures (based on position operators $\sin\varphi$, $\cos\varphi$, or $\exp(\mathrm{i}2\varphi)$) appear to be unsatisfactory. The new measures resort to multiplication angle operator variance (see equations (11), (12) and (17)) and obey uncertainty relations of the Schrödinger–Robertson type (with appropriate generalizations of the notions of covariance and mean commutator for the angle and angular momentum observables). The first two measures are defined as the arithmetic mean of the angle variance or as the minimal value of the variance within any 2π length interval. The latter appears to coincide with the angle variance, calculated by integration from $\varphi_c - \pi$ to $\varphi_c + \pi$, where φ_c is the centre of the wave packet, defined appropriately. The values of these measures are determined solely by the wavefunction $\psi(\varphi)$ of the particle.

The position and the angular momentum uncertainty measures can be used to define delocalization measures on the phase space (here it is a cylinder $S^1 \times \mathbb{R}$). Such measures can be defined as a sum or as a product of position uncertainties ${}_i\Delta^2\varphi$, $i = a, b, c$, and angular momentum variance $\Delta^2 l_z$. These possibilities stem from equations (9) and (14). From (9) and (14) we also derive the uncertainty relations ($\Delta^2 l_z = (\Delta l_z)^2$)

$$\Delta^2 l_z + {}_i\Delta^2\varphi \geqslant 2\big|_i\left(\mathrm{Im}\, G_{\varphi l_z}\right)\big| \qquad i = a, b, c. \tag{20}$$

The counterpart of this inequality on the real line is $\Delta^2 x + \Delta^2 p_x \geqslant 1$, which is minimized in the canonical CS $|\alpha\rangle$ *only* [12]. There are no periodic wavefunctions on the circle that precisely minimize (20). Calculations show that they are approximately minimized in the CS on the circle $|\xi\rangle$ [5, 6, 8]: in $|\xi\rangle$ the sum ${}_c\Delta^2 l_z + {}_c\Delta^2\varphi$ attains the minimal value, which is very close to 1. In this sense $|\xi\rangle$ are most localized states in the phase space. Let us note that in $|\xi\rangle$ one also has $\Delta^2(\hat{l}_z) + \Delta^2(\hat{\varphi}) = 1$ [3].

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